

DILUTE BOSE-EINSTEIN CONDENSATE IN A TRAP: CHARACTERISTIC LENGTHS AND CRITICAL VELOCITIES

ALEXANDER L. FETTER

*Departments of Physics and Applied Physics, Stanford University
Stanford, CA 94305-4060, USA
E-mail: fetter@leland.stanford.edu*

The Bogoliubov approximation and the Gross-Pitaevskii equation characterize the effect of repulsive interactions on a dilute ideal Bose-Einstein gas in a spherical harmonic trap. For large N , the interactions expand the condensate relative to an ideal Bose gas; both the speed of sound and critical angular velocity Ω_{c1} for creation of a quantized vortex depend crucially on the interparticle repulsion through the coherence length ξ .

1 Ideal Bose gas

An ideal Bose gas provides a valuable introduction to the physics of a real dilute Bose gas, for it emphasizes the fundamental role of the coherent condensate that contains a macroscopic number of particles in a single quantum state. The standard example treats N noninteracting bosons in a volume V with periodic boundary conditions and uniform density $n = N/V$. In the classical limit, the thermal de Broglie wavelength $\lambda_T \equiv (2\pi\hbar^2/mk_BT)^{1/2}$ is much shorter than the interparticle spacing $l \sim n^{-1/3}$. As the temperature falls, however, the thermal de Broglie wavelength grows, and the system eventually becomes degenerate when $\lambda_T \sim l$. Equivalently, Bose-Einstein condensation in a uniform ideal gas occurs at $T_c \sim \hbar^2 n^{2/3}/k_B m$.

The situation is somewhat different for an ideal Bose gas in a harmonic trap (taken as isotropic for simplicity) with $V_{\text{trap}}(r) = \frac{1}{2}m\omega_0^2 r^2$; the corresponding oscillator length $d_0 = (\hbar/m\omega_0)^{1/2}$ characterizes the size of the (Gaussian) ground state. In the classical limit ($k_B T \gg \hbar\omega_0$), the density follows the Boltzmann distribution $n_{cl}(r) \propto \exp[-V_{\text{trap}}(r)/k_B T]$, which can be rewritten as $\exp(-r^2/2R_T^2)$, with $R_T \equiv d_0(k_B T/\hbar\omega_0)^{1/2}$ the classical thermal radius of the trapped gas (note that $R_T \gg d_0$).

To estimate the temperature T_c for the onset of Bose-Einstein condensation in a trap, use the same expression with $n \sim N/R_T^3$, so that $k_B T_c \sim \hbar^2 N^{2/3}/mR_T^2 \sim \hbar^2 \omega_0^2 N^{2/3}/k_B T_c$; equivalently, $k_B T_c \sim N^{1/3} \hbar\omega_0 \gg \hbar\omega_0$. Since $d_0 \ll R_T$, the condensate forms a narrow spike of width d_0 superimposed on the smooth background of width R_T , as is seen clearly in the original experiment with a few thousand ^{87}Rb atoms.¹ More recent experiments have condensed millions of atoms, with typical parameters $N \approx 5 \times 10^6$, $\omega_0/2\pi \approx 120$ Hz, and $d_0 \approx 1.9 \mu\text{m}$ for sodium atoms.²

2 Effect of Repulsive Interactions

The interparticle spacing is typically large compared to the range of the atomic interactions, so that the interparticle potential can be taken as $U(r) \approx U_0 \delta(\mathbf{r})$, with $U_0 = 4\pi\hbar^2/m$ and a the s -wave scattering length ($a \approx 2.75$ nm for sodium atoms). The Bogoliubov approximation³ assumes that nearly all particles are in the

condensate; at zero temperature, the system forms a dilute Bose gas with $na^3 \ll 1$.

2.1 Quasiparticles and the Coherence Length

A uniform Bose gas at $T = 0$ K has only two characteristic single-particle energies: the kinetic energy $\epsilon_k = \hbar^2 k^2 / 2m$ and the “Hartree” energy

$$V_H(\mathbf{r}) = \int d^3 r' U(\mathbf{r} - \mathbf{r}') n(\mathbf{r}') = U_0 n = 4\pi a \hbar^2 n / m. \quad (1)$$

Bogoliubov showed that a quasiparticle with wave vector \mathbf{k} has an energy³

$$E_k = \sqrt{2V_H \epsilon_k + \epsilon_k^2} \approx \begin{cases} \hbar s k, & \text{for } k \rightarrow 0 \text{ (phonons),} \\ \epsilon_k = \hbar^2 k^2 / 2m, & \text{for } k \rightarrow \infty \text{ (free particles),} \end{cases} \quad (2)$$

where $s^2 = U_0 n / m = 4\pi a \hbar^2 n / m^2$ is the squared speed of sound. The quasiparticle is a linear combination of a particle and a hole. For $k \rightarrow 0$, it is an equal particle-hole admixture, but it becomes a pure particle as $k \rightarrow \infty$, with the cross-over occurring at $k \approx \xi^{-1}$, where $\xi = (8\pi a n)^{-1/2}$ defines the “coherence” length. Note that the speed of sound can be rewritten as $s = \kappa / 2\pi\sqrt{2}\xi$, where $\kappa = h/m$ is the quantum of circulation. When $na^3 \ll 1$, it is easy to verify that $\xi \gg l \gg a$.

2.2 Radius of Interacting Condensate in a Trap

The presence of the trap introduces a third characteristic energy, and the Gross-Pitaevskii (GP) equation^{4,5} for the nonuniform condensate wave function $\Psi(\mathbf{r})$ has the form

$$(T + V_{\text{trap}} + V_H)\Psi = \mu\Psi, \quad (3)$$

where $T = -\hbar^2 \nabla^2 / 2m$ is the kinetic energy operator, $V_{\text{trap}}(\mathbf{r}) = \frac{1}{2} \hbar \omega_0 (r/d_0)^2$ is the trap potential energy, $V_H(\mathbf{r}) = U_0 n(\mathbf{r}) = 4\pi a \hbar^2 n(\mathbf{r}) / m$ is the Hartree potential energy of one particle with the remaining particles, and μ is the chemical potential.

For very low density, the coherence length ξ exceeds the trap size d_0 , and the system acts like an ideal Bose gas. As N increases, however, ξ shrinks; when ξ becomes comparable with d_0 , the repulsive interactions begin to predominate, and the self-consistent radius R_0 of the condensate expands beyond d_0 .

To estimate the actual radius, note that the kinetic energy per particle is of order $T \sim \hbar^2 / m R_0^2 \sim \hbar \omega_0 / R^2$, where $R \equiv R_0 / d_0$ is the dimensionless radius of the condensate. Similarly, the order of magnitude of the remaining single-particle energies are $V_{\text{trap}} \sim \hbar \omega_0 R^2$, and

$$V_H \sim U_0 n \sim \frac{\hbar^2 a}{m} n \sim \frac{\hbar^2 a}{m} \frac{N}{R_0^3} \sim \frac{\hbar \omega_0}{R^3} \frac{Na}{d_0}, \quad (4)$$

where the dimensionless parameter $\eta \equiv Na/d_0$ characterizes the strength of the interactions. For $\eta \ll 1$, the system is effectively ideal, but for $\eta \gg 1$, the interactions become crucial (nevertheless, the system remains dilute with $a \ll l$ for all practical trapped condensates). If $\langle \dots \rangle \equiv \int dV \Psi^* \dots \Psi$ denotes a condensate ground-state expectation value, the total energy $E = \langle T + V_{\text{trap}} + \frac{1}{2} V_H \rangle$ is of order $\sim N \hbar \omega_0 (R^{-2} + R^2 + \eta R^{-3})$, and R is determined by minimizing E .

For small η , the minimum energy occurs at $R \approx 1$, and the condensate radius is just d_0 , with $\mu \approx \frac{3}{2}\hbar\omega_0$ (the ground-state energy of the isotropic oscillator). For large η , in contrast, the kinetic energy is negligible, and the minimum total energy occurs for $R^5 \sim \eta$. In this limit, a detailed calculation⁶ shows that $\mu = \frac{1}{2}\hbar\omega_0 R^2$ and that $R \equiv R_0/d_0 = (15\eta)^{1/5}$, which quantifies the expansion of the condensate relative to d_0 . For parameters appropriate to the MIT experiments,² the dimensionless condensate radius is $R \approx 10.2$, so that $R_0 \approx 19.3 \mu\text{m}$.

The resulting ‘‘Thomas-Fermi’’ (TF) approximation neglects the kinetic energy entirely, and the GP Eq. (3) then determines the condensate density by the condition $V_{\text{trap}}(\mathbf{r}) + V_H(\mathbf{r}) = \mu$, showing that the resulting density profile is parabolic. The corresponding central density $n(0) = R_0^2/8\pi ad_0^4 = (15/8\pi) N/R_0^3$ serves to define both the speed of sound $s^2 = 4\pi a\hbar^2 n(0)/m^2$ and the coherence length $\xi^2 = 1/8\pi an(0)$ for a trapped condensate; as a corollary, the relation $\xi R_0 = d_0^2$ implies that d_0 is the geometric mean of ξ and R_0 . The previous expression $s = \kappa/2\pi\sqrt{2}\xi$ then shows that the speed of sound in a large trapped condensate increases linearly with R_0 , reflecting the increased density. For the MIT experiments,² these relations yield $n(0) \approx 4 \times 10^{20} \text{ m}^{-3}$, $s \approx 1.04 \text{ cm/s}$, and $\xi \sim 0.187 \mu\text{m}$.

In the Bogoliubov approximation, most of the particles remain in the condensate, which requires³ $[n(0)a^3]^{1/2} \ll 1$. For a trapped condensate at $T = 0 \text{ K}$, this condition of small total noncondensate number holds for $R_0 a/d_0^2 \ll 1$. When this dimensionless ratio becomes comparable with 1, however, the repulsive interactions are so strong that the total noncondensate number becomes of order N (this situation occurs in liquid helium, even at $T = 0 \text{ K}$). Comparison with the TF expression $\eta = Na/d_0 \sim R^5$ shows that the Bogoliubov approximation fails for a trapped condensate when $N \sim (d_0/a)^6$, which is far larger than any current experimental value (for the MIT trap with sodium atoms, this limit merely requires that $N \ll 10^{15}$).

3 Critical Linear and Angular Velocities

Landau’s original explanation of superfluidity involved the critical velocity v_c ; it is the speed at which a moving macroscopic impurity can create quasiparticles and thus lose energy. A simple analysis shows that v_c is the minimum value of the ratio ω_k/k , where ω_k is the dispersion relation for the quasiparticles. From this perspective, a uniform ideal Bose gas has zero critical velocity, for the dispersion relation is simply $\hbar k^2/2m$. In contrast, Eq. (2) shows that a uniform dilute interacting Bose gas indeed has a nonzero critical velocity, with $v_c = s \propto a^{1/2}$, arising from the presence of the repulsive interactions.

3.1 Critical Angular Velocities in a Trapped Condensate

For a dilute trapped condensate, it is natural to take the speed of sound s as the critical velocity. It is impractical to shoot an impurity through the condensate, but the equatorial speed of a rotating condensate can serve to define a corresponding critical angular velocity $\Omega_c = s/R_0 = \hbar/md_0^2\sqrt{2} = \omega_0/\sqrt{2}$. Thus a large vortex-free condensate rotating at an angular velocity $\Omega \leq \Omega_c$ should remain superfluid (this expression also indicates that the frequencies of low-lying compressional modes in

a trapped condensate are independent of the radius and of order ω_0).

The low-lying hydrodynamic modes of a large spherical condensate have the dispersion relation $\omega_{nl} = \omega_0[l + n(2n + 2l + 3)]^{1/2}$, where n is the radial quantum number and l is the orbital-angular-momentum quantum number.⁷ For fixed l , the radial wavenumber k is $\approx n/R_0$, and a corresponding more precise Landau critical angular velocity is $\sqrt{2}\omega_0$ (this value occurs as $k \rightarrow \infty$).

It is helpful to review the effect of rotation $\mathbf{\Omega} = \Omega\hat{z}$ on a sample of liquid helium. If the fluid is normal, the microscopically rough walls bring it into solid-body rotation $\mathbf{v}_{\text{sb}} = \mathbf{\Omega} \times \mathbf{r}$, where \mathbf{r} is the distance from the axis of rotation; this flow has uniform vorticity, with $\nabla \times \mathbf{v}_{\text{sb}} = 2\mathbf{\Omega}$. In the rotating frame, the walls are stationary, and the relevant zero-temperature thermodynamic function is the “free energy” $F = E - \mathbf{\Omega} \cdot \mathbf{L}$ where E is the total ground-state energy and \mathbf{L} is the total ground-state angular momentum. For a circular cylinder of radius R_0 , the free energy of a state with one vortex on the symmetry axis becomes lower than that with no vortex at a lower critical angular velocity $\Omega_{c1} = (\kappa/2\pi R_0^2) \ln(R_0/\xi)$, where ξ represents the vortex core radius (ξ is a few atomic diameters for superfluid ^4He).

Assuming that a similar expression holds for the creation of a vortex in a large trapped condensate,^{6,8} the lower critical angular velocity is smaller than the trap frequency $\omega_0 = \kappa/2\pi d_0^2$ by a factor of order $(\xi/R_0) \ln(R_0/\xi) \ll 1$. In addition to the compressional modes with frequencies of order ω_0 , the presence of a vortex line introduces new dynamical degrees of freedom associated with “vortex waves.” At long wavelengths ($k\xi \ll 1$), the classical vortex-wave dispersion relation $\omega_k \approx (\kappa k^2/4\pi) \ln(1/k\xi)$ immediately suggests low-lying normal modes with $k \sim R_0^{-1}$ and frequencies of order Ω_{c1} , which might serve to signal their presence.

3.2 Analogy with Type-II Superconductors

These results for Ω_c and Ω_{c1} are very similar to the thermodynamic critical field $H_c = \Phi_0/2\pi\sqrt{2}\lambda\xi$ and lower critical field $H_{c1} \approx (\Phi_0/2\pi\lambda^2) \ln(\lambda/\xi)$, where $\Phi_0 = h/2e$ is the flux quantum, λ is the penetration length, and ξ is vortex core radius (also the superconducting coherence length).⁹ A uniform bulk superconductor becomes unstable with respect to normal metal at the field H_c , and the formation of quantized flux lines (vortices) becomes favorable at the field H_{c1} . In a type-II superconductor (one with $\lambda > \xi/\sqrt{2}$), the thermodynamic instability at H_c is preempted by vortex formation at $H_{c1} < H_c$.

The electromagnetic currents in a charged superfluid cut off the logarithmic intervortex interaction potential, screening it exponentially beyond the penetration length λ . This quantity diverges as the charge on each particle tends to zero, and the corresponding “screening” length in a neutral superfluid becomes either the radius of the container or the intervortex separation, whichever is smaller.

In addition, a type-II superconductor ultimately becomes normal at the upper critical field $H_{c2} = \Phi_0/2\pi\xi^2$, roughly when the vortex cores overlap. Unfortunately, the corresponding $\Omega_{c2} = \kappa/2\pi\xi^2$ is unattainably large in superfluid ^4He , and mechanical instability for $\Omega \geq \omega_0$ may also render it unobservable in a rotating dilute trapped condensate (in this case, a trapped condensate in equilibrium could contain only relatively few vortices).

3.3 Non-Axisymmetric Rotating Traps

The magnetic field that confines a dilute atomic Bose condensate acts simply as a potential $V_{\text{trap}}(\mathbf{r})$; this situation differs greatly from the microscopically rough walls of a container for superfluid helium. Thus, a “rotating trap” is meaningful only to the extent that it is nonaxisymmetric, for the rotating time-dependent potential pushes the condensate, setting it into motion.

In contrast to the case of an axisymmetric potential such as a circular cylinder, the free energy $E - \Omega L$ of an irrotational vortex-free state in a nonaxisymmetric trap decreases with increasing Ω (like $-\frac{1}{2}I\Omega^2$, where I is an effective moment of inertia). In the limit of large distortion, I can approach that of solid-body rotation. The negative free energy of the vortex-free state typically delays the onset of vortex formation, for the vorticity localized in the vortex cores becomes less necessary to “mimic” the uniform vorticity of \mathbf{v}_{sb} . This effect is readily verified in simple cases. For a uniform superfluid in a long rotating elliptic cylinder with semiaxes a and b ,¹⁰ the preceding expression $\Omega_{c1} = (\kappa/2\pi b^2) \ln(b/\xi)$ applies if $b = a$, but the corresponding lower critical angular velocity $\Omega_{c1} \approx (\kappa/4\pi b^2) \ln(b/\xi)$ for $b \ll a$ can become significantly larger.

The GP Eq. (3) provides a basis for analyzing a rotating trap, where the confining potential $V_{\text{trap}}(\mathbf{r})$ is stationary in the rotating frame (\mathbf{r} now denotes the coordinate in the rotating frame). The free energy is given by $E - \Omega L_z$, where E is the ground-state energy (assuming $T = 0$ K for simplicity). The condensate wave function $\Psi = |\Psi|e^{iS}$ can be expressed as a magnitude $|\Psi| = n^{1/2}$ and a phase S that determines the superfluid velocity $\mathbf{v} = (\hbar/m)\nabla S$. The TF limit of a large condensate neglects the spatial variation of n , and the free energy becomes

$$F \approx \int d^3r \left(\frac{1}{2}mnv^2 + nV_{\text{trap}} + \frac{1}{2}U_0n^2 - mn\Omega\hat{z} \cdot \mathbf{r} \times \mathbf{v} \right). \quad (5)$$

In a singly connected container, \mathbf{v} can be written as a sum of two contributions: $\mathbf{v}_\Omega \propto \Omega$ arising from the moving walls and \mathbf{v}_κ arising from the vortices with circulation κ . Correspondingly, F separates into a (vortex-free) contribution F_Ω associated solely with \mathbf{v}_Ω and $F_\kappa = \int d^3r mn\mathbf{v}_\Omega \cdot \mathbf{v}_\kappa + \frac{1}{2}mnv_\kappa^2 - mn\Omega\hat{z} \cdot \mathbf{r} \times \mathbf{v}_\kappa$ that also depends on the angular velocity Ω , both explicitly and through \mathbf{v}_Ω . The critical angular velocity Ω_{c1} for vortex creation occurs when F_κ first vanishes.

The integral for F provides a variational expression in the rotating frame; as an approximate trial function for the phase, take $S = -(\hbar/m)\Phi$, where Φ is the classical velocity potential for a uniform irrotational fluid (conventionally defined so that $\mathbf{v} = -\nabla\Phi$). This velocity field is automatically irrotational apart from the vortex cores and satisfies the condition that its normal derivative match the normal velocity of the rotating boundary. In addition, $\nabla \cdot \mathbf{v} = 0$, which is the appropriate TF limit of the continuity equation $\nabla \cdot (n\mathbf{v}) = 0$. The actual trapped-condensate density differs from that for $\Omega = 0$ only because of the Coriolis and centrifugal forces of order $m\Omega \times \mathbf{v}$ and $m\Omega \times (\Omega \times \mathbf{r})$, respectively. Each of these is small compared to the force of the harmonic trap so long as $\Omega \ll \omega_0$, in which case it is permissible to neglect the change in density (apart from the formation of the small vortex core of radius $\approx \xi$).

As a result, the free energy F_Ω for an irrotational condensate in a rotating trap has the same form as that for a uniform incompressible irrotational fluid, but with

the particle density taken from the solution of the GP equation. Since the actual density varies slowly in the TF limit, F_Ω differs from the classical expression only through factors of order unity, representing various moments of the actual density. Similarly, the additional free energy F_κ arising from the presence of a vortex line also differs from the classical expression for a uniform incompressible fluid only by factors of order unity, because the TF density profile cuts off the logarithmic divergence in the kinetic energy at a distance of order ξ , which can be identified with the classical vortex radius. Consequently, the classical analysis¹⁰ for uniform fluid in a rotating elliptical cylinder provides a qualitative guide to the corresponding problem of vortex nucleation in a rotating elliptical trap.

This correspondence remains valid for a multiply connected container, when the velocity field \mathbf{v} includes an additional contribution \mathbf{v}_Γ arising from the quantized circulation $\Gamma = j\kappa$ around the various internal boundaries (here, j is an integer). For example, consider an incompressible fluid in an asymmetric annular region between two nonconcentric cylinders (outer radius R_2 and inner radius $R_1 < R_2$), with the centers displaced a distance $d \leq R_2 - R_1$. If the system rotates in equilibrium at angular velocity Ω around the symmetry axis of the inner cylinder, the critical angular velocity Ω_κ for the creation of vortex-free quantized circulation κ around the inner boundary can be evaluated for arbitrary d . In the symmetrical limit ($d \rightarrow 0$), the result $\Omega_\kappa = (\kappa/2\pi)(R_2^2 - R_1^2)^{-1} \ln(R_2/R_1)$ is similar to that for a cylinder; in the small-gap limit, however, the resulting expression $\Omega_\kappa \approx \kappa/4\pi R_1 R_2$ has no logarithmic factor. These expressions can serve to estimate the corresponding quantities for a trapped toroidal condensate (created, for example, by piercing the condensate with an off-axis laser beam).¹¹ The circulation-induced deformation of the condensate might serve to detect the presence of a persistent current.

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